8 Vector norms

1. Consider vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. Use Pitagorian theorem and calculate lengths $\|u\|$ and $\|v\|$ of vectors $u \in \mathbb{R}^2$, $v \in \mathbb{R}^3$, and geometrically illustrate/show length of these vectors.

> **Euclidean Vector Norm** For a vector $\boldsymbol{x}$, the *euclidean norm* of $\boldsymbol{x}$ is defined to be
>
> - $\|\boldsymbol{x}\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2} = \sqrt{\boldsymbol{x}^\top \boldsymbol{x}}$ whenever $\boldsymbol{x} \in \mathbb{R}^n$,
> - $\|\boldsymbol{x}\| = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} = \sqrt{\boldsymbol{x}^* \boldsymbol{x}}$ whenever $\boldsymbol{x} \in \mathbb{C}^n$.

2. Find euclidean norm of $\boldsymbol{u} = (0, -1, 2, -2, 4)^\top$ and $\boldsymbol{v} = (i, 2, 1-i, 0, 1+i)^\top$. [$\|\boldsymbol{u}\| = 5$, $\|\boldsymbol{v}\| = 3$]

3. (a) Using the euclidean norm, describe the solid ball in $\mathbb{R}^n$ centered at the origin with unit radius. (b) Describe a solid ball centered at the point $c = (\xi_1, \xi_2, ..., \xi_n)$ with radius ρ.

> **Standard Inner Product** The scalar terms defined by
>
> $$\boldsymbol{x}^\top \boldsymbol{y} = \sum_{i=1}^n x_i y_i \in \mathbb{R} \quad \text{and} \quad \boldsymbol{x}^* \boldsymbol{y} = \sum_{i=1}^n \overline{x_i} y_i \in \mathbb{C}$$
>
> are called the *standard inner products* for $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively.

> **Cauchy-Bunyakovskii-Schwarz (CBS) Inequality**
>
> $$|\boldsymbol{x}^* \boldsymbol{y}| \leq \|\boldsymbol{x}\| \, \|\boldsymbol{y}\| \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n.$$
>
> Equality holds if and only if $\boldsymbol{y} = \alpha \boldsymbol{x}$ for $\alpha = \boldsymbol{x}^* \boldsymbol{y} / \boldsymbol{x}^* \boldsymbol{x}$.

> **Triangle Inequality**
>
> $$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\| \quad \text{for every } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n.$$

The Cauchy-Bunyakovskii-Schwarz (CBS) inequality[6] is one of the most important inequalities in mathematics. It relates inner product to norm.

4. Consider the euclidean norm with $\boldsymbol{u} = (2, 1, -4, -2)^\top$ and $\boldsymbol{v} = (1, -1, 1, -1)$. (a) Determine the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$. (b) Verify that the triangle inequality holds for $\boldsymbol{u}$ and $\boldsymbol{v}$. (c) Verify that the CBS inequality holds for $\boldsymbol{u}$ and $\boldsymbol{v}$.

5. **(Backward Triangle Inequality.)** Show that $|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\|$.

> **p-Norms** For $p \geq 1$, the p-norm of $x \in \mathbb{C}^n$ is defined as $\|\boldsymbol{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$.

6. To get a feel for the 1-, 2-, and ∞-norms, it helps to know the shapes and relative sizes of the unit p-spheres $\mathcal{S}_p = \{\boldsymbol{x} \mid \|x\|_p = 1\}$ for $p = 1, 2, \infty$. In space $\mathbb{R}^3$ give illustration of the unit 1-, 2-, and ∞-spheres.

7. Find 1-, 2-, and ∞-norms of $\boldsymbol{x} = (2, 1, -4, -2)^\top$ and $\boldsymbol{y} = (1+i, 1-i, 1, 4i)^\top$.

> **General Vector Norms** A *norm* for a real or complex vector space $\mathcal{V}$ is a function $\|\star\|$ mapping $\mathcal{V}$ into $\mathbb{R}$ that satisfies the following conditions.
>
> $$\|\boldsymbol{x}\| \geq 0 \quad \text{and} \quad \|\boldsymbol{x}\| = 0 \Longleftrightarrow \boldsymbol{x} = 0$$
>
> $$\|\alpha \boldsymbol{x}\| = |\alpha| \|\boldsymbol{x}\| \quad \text{for all scalars } \alpha,$$
>
> $$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|.$$

8. Show that $(\alpha_1 + \alpha_2 + ... + \alpha_n)^2 \leq n(\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2)$ for $\alpha_i \in \mathbb{R}$.

9. If $x, y \in \mathbb{R}^n$ such that $\|\boldsymbol{x} - \boldsymbol{y}\|_2 = \|\boldsymbol{x} + \boldsymbol{y}\|_2$, what is $\boldsymbol{x}^\top \boldsymbol{y}$?

10. Explain why $\|\boldsymbol{x} - \boldsymbol{y}\| = \|\boldsymbol{x} + \boldsymbol{y}\|$ is true for all norms.

InC: 1, 4, 6, 9. HW: few problems from the web page `http://osebje.famnit.upr.si/~penjic/linearnaAlgebra/`.

[6]The Cauchy-Bunyakovskii-Schwarz inequality is named in honor of the three men who played a role in its development. The basic inequality for real numbers is attributed to Cauchy in 1821, whereas Schwarz and Bunyakovskii contributed by later formulating useful generalizations of the inequality involving integrals of functions.

Augustin-Louis Cauchy (1789-1857) was a French mathematician who is generally regarded as being the founder of mathematical analysis-including the theory of complex functions. Although deeply embroiled in political turmoil for much of his life (he was a partisan of the Bourbons), Cauchy emerged as one of the most prolific mathematicians of all time. He authored at least 789 mathematical papers, and his collected works fill 27 volumes-this is on a par with Cayley and second only to Euler. It is said that more theorems, concepts, and methods bear Cauchy's name than any other mathematician.

Victor Bunyakovskii (1804-1889) was a Russian professor of mathematics at St. Petersburg, and in 1859 he extended Cauchy's inequality for discrete sums to integrals of continuous functions. His contribution was overlooked by western mathematicians for many years, and his name is often omitted in classical texts that simply refer to the Cauchy-Schwarz inequality.

Hermann Amandus Schwarz (1843-1921) was a student and successor of the famous German mathematician Karl Weierstrass at the University of Berlin. Schwarz independently generalized Cauchy's inequality just as Bunyakovskii had done earlier.

9 Inner-product spaces

General Inner Product An *inner product* on a real (or complex) vector space $\mathcal{V}$ is a function that maps each ordered pair of vectors $\boldsymbol{x}$, $\boldsymbol{y}$ to a real (or complex) scalar $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ such that the following four properties hold.

- $\langle \boldsymbol{x}, \boldsymbol{x}\rangle$ is real with $\langle \boldsymbol{x}, \boldsymbol{x}\rangle \geq 0$, and $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0$ if and only if $\boldsymbol{x} = 0$,
- $\langle \boldsymbol{x}, \alpha\boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ for all scalars α,
- $\langle \boldsymbol{x}, \boldsymbol{y}+\boldsymbol{z}\rangle = \langle \boldsymbol{x}, \boldsymbol{y}\rangle + \langle \boldsymbol{x}, \boldsymbol{z}\rangle$,
- $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x}\rangle}$ (for real spaces, this becomes $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{y}, \boldsymbol{x}\rangle$).

Notice that for each fixed value of $\boldsymbol{x}$, the second and third properties say that $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ is a linear function of $\boldsymbol{y}$.
Any real or complex vector space that is equipped with an inner product is called an *inner-product space*.

1. For $x = (x_1, x_2, x_3)^\top$ and $y = (y_1, y_2, y_3)^\top$, determine which of the following are inner products for $\mathbb{R}^3$. (a) $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = x_1y_1 + x_3y_3$,
(b) $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = x_1y_1 - x_2y_2 + x_3y_3$,
(c) $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 2x_1y_1 + x_2y_2 + 4x_3y_3$,
(d) $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2$.

General CBS Inequality If $\mathcal{V}$ is an inner-product space, and if we set $\|\star\| = \sqrt{\langle\star,\star\rangle}$, then

$$|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\| \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}.$$

Equality holds if and only if $\boldsymbol{y} = \alpha\boldsymbol{x}$ for $\alpha = \langle \boldsymbol{x}, \boldsymbol{y}\rangle/\|\boldsymbol{x}\|^2$.

Norms in Inner-Product Spaces If $\mathcal{V}$ is an inner-product space with an inner product $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$, then $\|\star\| = \sqrt{\langle\star,\star\rangle}$ defines a norm on $\mathcal{V}$.

2. (a) Show that the standard inner products, $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^\top\boldsymbol{y}$ for $\mathbb{R}^n$ and $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^\top\boldsymbol{y}$ for $\mathbb{R}^n$, each satisfy the four defining conditions above for a general inner product. (b) Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a nonsingular matrix. Show that $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^\top A^\top A\boldsymbol{y}$ is and inner product on $\mathbb{R}^n$. (c) Let $\mathcal{V}$ denote the vector space of real-valued continuous functions defined on the interval (a, b). Show that $\langle f, g\rangle = \int_a^b f(t)g(t)\,\mathrm{d}t$ is an inner product space on $\mathcal{V}$.

3. For $A, B \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, determine is the following product
$\langle A, B\rangle = \mathrm{trace}(A^\top B) := \sum_{i=1}^n (A^\top B)_{ii}$
an inner products for $\mathbb{R}^3$.

4. Describe the norms that are generated by the inner products in Exercises 2 and 3.

5. To illustrate the utility of the ideas presented above, consider the proposition

$$\mathrm{trace}(A^\top B)^2 \leq \mathrm{trace}(A^\top A)\,\mathrm{trace}(B^\top B)$$

for all $A, B \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. How would you know to formulate such a proposition and, second, how do you prove it?

Since each inner product generates a norm by the rule $\|\star\| = \sqrt{\langle\star,\star\rangle}$, it's natural to ask if the reverse is also true. That is, for each vector norm $\|\star\|$ on a space $\mathcal{V}$, does there exist a corresponding inner product on $\mathcal{V}$ such that $\sqrt{\langle\star,\star\rangle} = \|\star\|$? If not, under what conditions will a given norm be generated by an inner product? These are tricky questions, and it took the combined efforts of Maurice R. Fréchet[7] (1878–1973) and John von Neumann (1903–1957) to provide the answer.

Parallelogram Identity For a given norm $\|\star\|$ on a vector space $\mathcal{V}$, there exists an inner product on $\mathcal{V}$ such that $\langle\star,\star\rangle = \|\star\|^2$ if and only if the *parallelogram identity*

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2)$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$.

6. Except for the euclidean norm, is any other vector p-norm generated by an inner product?

InC: 1, 2, 5, 6. HW: 3 + see `http://osebje.famnit.upr.si/~penjic/linearnaAlgebra/`.

[7] Maurice René Fréchet began his illustrious career by writing an outstanding Ph.D. dissertation in 1906 under the direction of the famous French mathematician Jacques Hadamard in which the concepts of a metric space and compactness were first formulated. Fréchet developed into a versatile mathematical scientist, and he served as professor of mechanics at the University of Poitiers (1910–1919), professor of higher calculus at the University of Strasbourg (1920–1927), and professor of differential and integral calculus and professor of the calculus of probabilities at the University of Paris (1928–1948).

Born in Budapest, Hungary, John von Neumann was a child prodigy who could divide eightdigit numbers in his head when he was only six years old. Due to the political unrest in Europe, he came to America, where, in 1933, he became one of the six original professors of mathematics at the Institute for Advanced Study at Princeton University, a position he retained for the rest of his life. During his career, von Neumann's genius touched mathematics (pure and applied), chemistry, physics, economics, and computer science, and he is generally considered to be among the best scientists and mathematicians of the twentieth century.